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WS 19/20
Sheet 15

Exercises for Stochastic Processes

Tutorial exercises:

T1. We consider the graph G with vertex set $V = \mathbb{Z}^2$ and nearest neighbour edges: $E := \{\{x, y\} : \|x - y\|_1 = 1\}$. In the Bernoulli site percolation model, we assign the spin 1 to a vertex with probability p and the spin 0 otherwise, independently for every vertex. To be precise, we consider the probability space

$$(\Omega, \mathcal{F}, \mathbb{P}_p) = \left(\{0,1\}^V, \, \sigma\big(\{\{\omega_v = 1\} \, : \, v \in V\}\big), \, \prod_v \mu_v^{(p)} \right),$$

where $\mu_v^{(p)}(\omega_v = 1) = p$ and $\mu_v^{(p)}(\omega_v = 0) = 1 - p$. We say that $0 \longleftrightarrow \infty$ if there is an infinite (self-avoiding) path starting in 0 of which every vertex has spin 1. We define the critcal value p_c as

$$p_c := \sup\{p \in [0,1] \, : \, \mathbb{P}_p(0 \longleftrightarrow \infty) = 0\}.$$

(a) Show that $\{0 \longleftrightarrow \infty\}$ is measurable.

(b) Show that $p_c > 0$.

(c) Show that $p_c < 1$.

T2. We consider the stochastic Ising model on $\mathbb{Z}^d$ with parameter $\beta \geq 0$.

(a) Show that this spin system is attractive.

(b) Show that $\varepsilon = 2$, and that

$$M = 2de^{2d\beta}(1 - e^{-2\beta}).$$

T3. Consider the one-dimensional stochastic Ising model with $\beta < 0$ and rates given by

$$c_\beta(x, \eta) := \exp\left(-\beta \sum_{y : y \sim x} (2\eta(x) - 1)(2\eta(y) - 1) \right).$$

Prove that this spin system is ergodic and find its invariant measure.

(Hint: consider the bijection $\phi : S \to S$ that flips the spin of all odd vertices and calculate $c_\beta(x, \phi(\eta))$.)